

Spatially-Coupled MacKay-Neal Codes with No Bit Nodes of Degree Two Achieve the Capacity of BEC

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Abstract—Obata et al. proved that spatially-coupled (SC) MacKay-Neal (MN) codes achieve the capacity of BEC. However, the SC-MN codes have many variable nodes of degree two and have higher error floors. In this paper, we prove that SC-MN codes with no variable nodes of degree two achieve the capacity of BEC.

I. INTRODUCTION

Felström and Zigangirov introduced spatially-coupled (SC) codes defined by sparse parity check matrix. SC codes are based on constitution method for convolutional LDPC codes [1]. Lantmaier et al. confirmed that regular SC LDPC codes achieve MAP threshold of original LDPC block codes by BP decoding in at least certain accuracy [2]. Kudekar et al. proved that SC codes achieve MAP threshold by BP decoding on binary erasure channel (BEC) [3] and binary symmetric channel [4].

Kasai et al. introduced SC MacKay-Neal (MN) codes, and showed that these codes with finite maximum degree achieve capacity of BEC by numerical experiment [5]. Obata et al. proved $(l, 2, 2)$ SC-MN codes achieve capacity [6]. It has been observed that $(l, 2, 2)$ SC-MN codes have many bit nodes of degree two. This leads to high error floors.

In this paper, we deal with $(l, 3, 3)$ SC-MN codes whose bit node degree is greater than two. We prove the codes achieve the capacity of BEC. The codes achieve Shannon limit $\epsilon^{\text{Sha}} = 1 - \frac{3}{l}$ for any $l \geq 3$.

II. BACKGROUND

A. MacKay-Neal Codes

(l, r, g) MN codes are multi-edge type (MET) LDPC codes defined by pair of multi-variables degree distributions (μ, ν) listed below.

$$\begin{aligned}\nu(\mathbf{x}; \epsilon) &= \frac{r}{l} x_1^l + \epsilon x_2^g, \\ \mu(\mathbf{x}) &= x_1^r x_2^g.\end{aligned}$$

In general, the recursion of density evolution of MET-LDPC codes on BEC is given by

$$y_j^{(t)} = 1 - \frac{\mu_j(\mathbf{1} - \mathbf{x}^{(t)}; 1 - \epsilon)}{\mu_j(\mathbf{1}; 1)}, \quad x_j^{(t+1)} = \frac{\nu_j(\mathbf{y}^{(t)}; \epsilon)}{\nu_j(\mathbf{1}; 1)},$$

where $x_j^{(t)}$ is probability of erasure message sent along edges of type j at the t -th decoding round. Therefore, density

evolution of (l, r, g) MN codes is

$$\begin{aligned}\mathbf{x}^{(t+1)} &= \mathbf{f}(\mathbf{g}(\mathbf{x}^{(t)}); \epsilon), \\ \mathbf{f}(\mathbf{x}; \epsilon) &= (x_1^{l-1}, \epsilon x_2^{g-1}), \\ \mathbf{g}(\mathbf{x}) &= (1 - (1 - x_1)^{r-1}(1 - x_2)^g, 1 - (1 - x_1)^r(1 - x_2)^{g-1}).\end{aligned}\tag{2}$$

B. Spatially-Coupled MacKay-Neal Codes

SC-MN codes of coupling number L and of coupling width w are defined by the Tanner graph constructed by the following process. First, at each section $i \in \mathbb{Z}$, place rM/l bit nodes of type 1 and M bits nodes of type 2. Bit nodes of type 1 and 2 are of degree l and g , respectively. Next, at each section $i \in \mathbb{Z}$, place M check nodes of degree $r + g$. Then, connect edges uniformly at random so that bit nodes of type 1 at section i are connected with check nodes at each section $i \in [i, \dots, i + w - 1]$ with rM/w edges, and bit nodes of type 2 at section i are connected with check nodes at each section $i \in [i, \dots, i + w - 1]$ with gM/w edges. Bits at section $i \notin [0, L - 1]$ are shorten. Bits of type 1 and 2 at section $i \in [0, L - 1]$ are punctured and transmitted, respectively. Rate of SC-MN codes R^{MN} is given by

$$R^{\text{MN}} = \frac{r}{l} + \frac{1 + w - 2 \sum_{i=0}^w (1 - (\frac{i}{w})^{r+g})}{L} = \frac{r}{l} \quad (L \rightarrow \infty).$$

C. Vector Admissible System and Potential Function

In this section, we define vector admissible systems and potential functions.

Definition 1. Define $\mathcal{X} \triangleq [0, 1]^d$, and $F : \mathcal{X} \times [0, 1] \rightarrow \mathbb{R}$ and $G : \mathcal{X} \rightarrow \mathbb{R}$ as functionals satisfying $G(\mathbf{0}) = 0$. Let \mathbf{D} be a $d \times d$ positive diagonal matrix. Consider a general recursion defined by

$$\mathbf{x}^{(t+1)} = \mathbf{f}(\mathbf{g}(\mathbf{x}^{(t)}); \epsilon),$$

where $\mathbf{f} : \mathcal{X} \times [0, 1] \rightarrow \mathcal{X}$ and $\mathbf{g} : \mathcal{X} \rightarrow \mathcal{X}$ are defined by $F'(\mathbf{x}; \epsilon) = \mathbf{f}(\mathbf{x}; \epsilon)\mathbf{D}$ and $G'(\mathbf{x}) = \mathbf{g}(\mathbf{x})\mathbf{D}$, where $F'(\mathbf{x}; \epsilon) \triangleq (\frac{\partial F(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial F(\mathbf{x})}{\partial x_n})$. Then the pair (\mathbf{f}, \mathbf{g}) defines a vector admissible system if

1. \mathbf{f}, \mathbf{g} are twice continuously differentiable,
2. $\mathbf{f}(\mathbf{x}; \epsilon)$ and $\mathbf{g}(\mathbf{x})$ are non-decreasing in \mathbf{x} and ϵ with respect to \preceq^1 ,

¹We say $\mathbf{x} \preceq \mathbf{y}$ if $x_i \leq y_i$ for all $1 \leq i \leq d$

3. $\mathbf{f}(\mathbf{g}(\mathbf{0}); \epsilon) = \mathbf{0}$ and $F(\mathbf{g}(\mathbf{0}); \epsilon) = 0$.

We say \mathbf{x} is a fixed point if $\mathbf{x} = \mathbf{f}(\mathbf{g}(\mathbf{x}); \epsilon)$.

It can be seen that the density evolution (\mathbf{f}, \mathbf{g}) of (l, r, g) MN codes given in (2) is a vector admissible system by choosing $F(\mathbf{x}; \epsilon), G(\mathbf{x})$ and \mathbf{D} as below, since this system (\mathbf{f}, \mathbf{g}) satisfies the condition in Definition 1.

$$\begin{aligned} F(\mathbf{x}; \epsilon) &= \frac{r}{l} x_1^l + \epsilon x_2^g, \\ G(\mathbf{x}) &= r x_1 + g x_2 + (1 - x_1)^r (1 - x_2)^g - 1, \\ \mathbf{D} &= \begin{pmatrix} r & 0 \\ 0 & g \end{pmatrix}. \end{aligned}$$

Definition 2 ([7, Def. 2]). We define the potential function $U(\mathbf{x}; \epsilon)$ of a vector admissible system (\mathbf{f}, \mathbf{g}) by

$$U(\mathbf{x}; \epsilon) \triangleq \mathbf{g}(\mathbf{x}) \mathbf{D} \mathbf{x}^T - G(\mathbf{x}) - F(\mathbf{g}(\mathbf{x}); \epsilon).$$

The potential function $U(x_1, x_2, \epsilon)$ of (l, r, g) MN codes is given by

$$\begin{aligned} U(x_1, x_2, \epsilon) &= 1 - \epsilon \left((1 - (1 - x_1)^r) (1 - x_2)^{g-1} \right)^g \\ &\quad - \frac{r}{l} \left(1 - (1 - x_1)^{r-1} (1 - x_2)^g \right)^l \\ &\quad - (1 - x_1)^r (1 - x_2)^g \left(1 + \frac{r x_1}{1 - x_1} + \frac{g x_2}{1 - x_2} \right). \end{aligned}$$

Definition 3 ([7, Def. 7]). Let $\mathcal{F}(\epsilon) \triangleq \{\mathbf{x} \in \mathcal{X} \setminus \{\mathbf{0}\} \mid \mathbf{x} = \mathbf{f}(\mathbf{g}(\mathbf{x}); \epsilon)\}$ be a set of non-zero fixed points for $\epsilon \in [0, 1]$. The potential threshold ϵ^* is defined by

$$\epsilon^* \triangleq \sup\{\epsilon \in [0, 1] \mid \min_{\mathbf{x} \in \mathcal{F}(\epsilon)} U(\mathbf{x}; \epsilon) > 0\}.$$

Let ϵ_s^* be threshold of uncoupled system defined in [7, Def. 6]. For ϵ such that $\epsilon_s^* < \epsilon < \epsilon^*$, we define energy gap $\Delta E(\epsilon)$ as

$$\Delta E(\epsilon) \triangleq \max_{\epsilon' \in [\epsilon, 1]} \inf_{\mathbf{x} \in \mathcal{F}(\epsilon')} U(\mathbf{x}; \epsilon').$$

We define the SC system of a vector admissible system.

Definition 4 ([7, Def. 9]). For a vector admissible system (\mathbf{f}, \mathbf{g}) , we define the SC system of coupling number L and coupling width w as

$$\begin{aligned} \mathbf{x}_i^{(t+1)} &= \frac{1}{w} \sum_{k=0}^{w-1} \mathbf{f} \left(\frac{1}{w} \sum_{j=0}^{w-1} \mathbf{g}(\mathbf{x}_{i+j-k}^{(t)}); \epsilon_{i-k} \right), \\ \epsilon_i &= \begin{cases} \epsilon, & i \in \{0, \dots, L-1\}, \\ 0, & i \notin \{0, \dots, L-1\}. \end{cases} \end{aligned}$$

If we define (\mathbf{f}, \mathbf{g}) as the density evolution for (l, r, g) MN codes in (2), the SC system gives the density evolution of SC-MN codes with coupling number L and coupling width w .

Next theorem states that if $\epsilon < \epsilon^*$ then fixed points of SC vector system converge towards $\mathbf{0}$ for sufficiently large w .

Theorem 1 ([7, Thm. 1]). Consider the constant $K_{\mathbf{f}, \mathbf{g}}$ defined in [7, Lem. 11]. This constant value depends only on (\mathbf{f}, \mathbf{g}) . If $\epsilon < \epsilon^*$ and $w > (dK_{\mathbf{f}, \mathbf{g}})/(2\Delta E(\epsilon))$, then the SC system

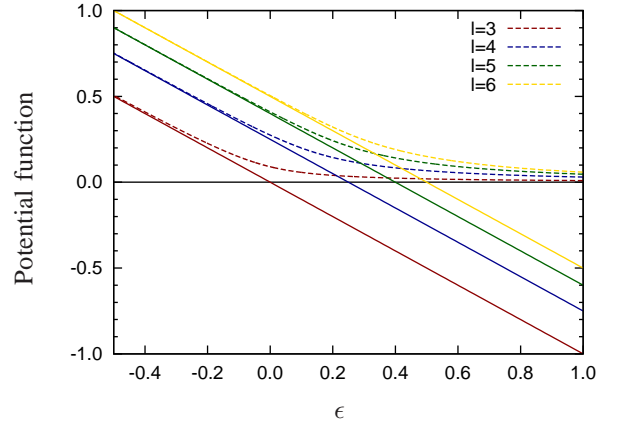


Fig. 1. Potential function $U(\mathbf{1}; \epsilon)$ and $U(\mathbf{x}(x_1); \epsilon(x_1))$ at the trivial fixed points (solid) and non-trivial fixed points (dashed) of $(l, 3, 3)$ MN codes for $l = 3, \dots, 6$

of (\mathbf{f}, \mathbf{g}) with coupling number L and coupling width w has a unique fixed point $\mathbf{0}$.

We will show that the potential threshold ϵ^* of $(l, r = 3, g = 3)$ MN codes is $1 - R^{\text{MN}} = 1 - 3/l$ for any $l \geq 3$. This is sufficient to show that $(l, 3, 3)$ SC-MN codes with sufficiently large w and L achieve the capacity of BEC under BP decoding.

III. PROOF OF ACHIEVING CAPACITY

In this section, we calculate the potential threshold ϵ^* of $(l, r = 3, g = 3)$ MN codes. To this end, we first investigate the set of fixed points $\mathcal{F}(\epsilon)$.

The density evolution recursion in (2) can be rewritten as

$$\begin{aligned} x_1^{(t+1)} &= (1 - (1 - x_1^{(t)})^{r-1} (1 - x_2^{(t)})^g)^{l-1}, \\ x_2^{(t+1)} &= \epsilon (1 - (1 - x_1^{(t)})^r (1 - x_2^{(t)})^{g-1})^{g-1}. \end{aligned}$$

Fixed points $(x_1, x_2; \epsilon)$ of density evolution with $x_1 = 0$ and $x_1 = 1$ are $(0, 0; \epsilon)$ and $(1, \epsilon; \epsilon)$, respectively. We define these fixed points as trivial fixed points and all other fixed points as non-trivial fixed points. All non-trivial fixed points $(x_1, x_2(x_1); \epsilon(x_1))$ can be parametrically described as

$$\begin{aligned} x_2(x_1) &= 1 - \left(\frac{1 - x_1^{\frac{l-1}{r-1}}}{(1 - x_1)^{r-1}} \right)^{\frac{1}{g}}, \\ \epsilon(x_1) &= \frac{x_2(x_1)}{(1 - (1 - x_1)^r (1 - x_2(x_1))^{g-1})^{g-1}}, \end{aligned}$$

with $x_1 \in (0, 1)$.

Next, we shall investigate the value of the potential function at the fixed points. The value of the potential functions at trivial fixed point $(1, \epsilon, \epsilon)$ is respectively given by

$$U(\mathbf{1}, \epsilon, \epsilon) = 1 - \frac{r}{l} - \epsilon.$$

Figure 1 draws the potential function of (l, r, g) MN codes at fixed points $\mathbf{x} \in \mathcal{F}(\epsilon)$. It appears that the potential function at non-trivial fixed points is always positive. We will prove this.

To be precise, the potential function of (l, r, g) MN codes for non-trivial fixed points satisfies

$$U(x_1, x_2(x_1), \epsilon(x_1)) > 0 \quad \text{for } x_1 \in (0, 1). \quad (4)$$

Our strategy of proof is as follows. First change the representation of (4) into a polynomial form by changing variables a few times. Then apply Sturm's theorem for smaller l and bound the polynomial for larger l .

We define $U(z) := U(x_1, x_2(x_1), \epsilon(x_1))|_{x_1=z^{l-1}}$. Obviously, to prove (4), it is sufficient to show $U(z) > 0$ for $z \in (0, 1)$.

$$U(z) = -\frac{3z^l}{l} + (1-z)(1-4z^{l-1}) + (1-z)^{1/3}(1-z^{l-1})^{-2/3} - 2(1-z)^{2/3}(1-z^{l-1})^{5/3}.$$

We use next lemma to eliminate fractional power in $U(z)$. The proof is given in Section IV-A.

Lemma 1. Define $H(u, z)$ as follows.

$$H(u, z) = \left(u + \frac{3z^l}{l} - (1-z)(1-4z^{l-1})\right)^3 + 6(1-z)(1-z^{l-1})\left(u + \frac{3z^l}{l} - (1-z)(1-4z^{l-1})\right) - (1-z)(1-z^{l-1})^{-2} + 8(1-z)^2(1-z^{l-1})^5.$$

Then, $H(0, z) < 0$ for $z \in (0, 1)$ implies $U(z) > 0$ for $z \in (0, 1)$.

Define $I(z) := \frac{l^3(1-z^{l-1})^2}{(1-z)z^2}H(0, z)$. Obviously, to prove $H(0, z) < 0$ for $z \in (0, 1)$, it is sufficient to prove $I(z) < 0$ for $z \in (0, 1)$. We see that $I(z)$ for $l \geq 3$ is a polynomial as follows.

$$I(z) = -l^3 + 27 \sum_{i=0}^{l-2} [z^{3l-2+i}(1-z^{l-1})] - 27l^2 z^{-2+2l}(1-4z^{l-1})(1-z^{l-1})^2 - 9lz^{-4+l}(1-z^{l-1})^2 \{(-3+z)z^2 + 16(-1+z)z^{2l} - 8(-1+z)z^{1+l}\} - l^3(1-z)z^{-9+l} \{8z^{6l} - 56z^{1+5l} + 2z^6(3+7z) + 8z^{2+4l}(13+8z) - 8z^{3+3l}(13+22z) + 4z^{4+2l}(21+43z) - z^{5+l}(41+73z)\}. \quad (5)$$

We prove $I(z) < 0$ for $3 \leq l < 165$ and $l \geq 165$ in the following lemmas. The proofs are given in Section IV-B and Section IV-C, respectively.

Lemma 2. For $3 \leq l < 165$, $I(z) < 0$ for $z \in (0, 1)$.

Lemma 3. For $l \geq 165$, $I(z) < 0$ for $z \in (0, 1)$.

Theorem 2. For any $l \geq 3$ and $\epsilon < \epsilon^{\text{Sha}} = 1 - \frac{3}{l}$, the unique fixed point of density evolution of $(l, 3, 3)$ SC-MN codes of coupling number L and coupling width w is 0 for sufficiently large w and L .

Proof: From (4), potential function for non-trivial fixed points is always positive. Therefore, from Definition 3 and potential function for trivial fixed point (3), $\epsilon^* = 1 - \frac{r}{l} = \epsilon^{\text{Sha}}$. From Theorem 1, for $\epsilon < \epsilon^{\text{Sha}}$, the unique fixed point of density evolution for $(l, 3, 3)$ SC-MN codes is 0. \square

The case with $l = 3$ implies rate one codes over BEC(0). Some might think this is not interesting. Nevertheless, we included the case with $l = 3$ for comprehensiveness.

IV. PROOF OF LEMMAS

A. Proof of Lemma 1

Partial derivative of $H(u, z)$ with respect to u gives

$$\frac{\partial H(u, z)}{\partial u} = 3\left(u + \frac{3z^l}{l} - (1-z)(1-4z^{l-1})\right)^2 + 6(1-z)(1-z^{l-1}) \geq 0. \quad (6)$$

Substituting $u = U(z)$ into $H(u, z)$ gives

$$\begin{aligned} H(U(z), z) &= ((1-z)^{1/3}(1-z^{l-1})^{-2/3} - 2(1-z)^{2/3}(1-z^{l-1})^{5/3})^3 \\ &\quad + 6(1-z)(1-z^{l-1})\{(1-z)^{1/3}(1-z^{l-1})^{-2/3} \\ &\quad - 2(1-z)^{2/3}(1-z^{l-1})^{5/3}\} \\ &\quad - (1-z)(1-z^{l-1})^{-2} + 8(1-z)^2(1-z^{l-1})^5 \\ &= (1-z)(1-z^{l-1})^{-2} - 8(1-z)^2(1-z^{l-1})^5 \\ &\quad - 6(1-z)(1-z^{l-1})\{(1-z)^{1/3}(1-z^{l-1})^{-2/3} \\ &\quad - 2(1-z)^{2/3}(1-z^{l-1})^{5/3}\} \\ &\quad + 6(1-z)(1-z^{l-1})\{(1-z)^{1/3}(1-z^{l-1})^{-2/3} \\ &\quad - 2(1-z)^{2/3}(1-z^{l-1})^{5/3}\} \\ &\quad - (1-z)(1-z^{l-1})^{-2} + 8(1-z)^2(1-z^{l-1})^5 \\ &= 0. \end{aligned} \quad (7)$$

From (6), $H(u, z)$ monotonically increasing with respect to u . From (7), $(u, z) = (U(z), z)$ is a root of $H(u, z) = 0$. Therefore $H(0, z) < 0$ for $z \in (0, 1)$ implies $U(z) > 0$ for $z \in (0, 1)$. \square

B. Proof of Lemma 2

From $I(0) = -l^3$ and $I(1) = -l^3$, we see that $z = 0, 1$ are not multiple roots of equation $I(z) = 0$. Let $I_1(z), \dots, I_m(z)$ be Sturm sequences of $I(x)$. Let $V(z)$ be the number of sign changes in the sequence. Table I lists sign changes of Sturm sequence $I_1(z), \dots, I_m(z)$ of $I(x)$ in (5) for $l = 3, \dots, 11$. $V(z)$ is the number of sign changes in the sequence. We see that $V(0) = V(1)$. We observed that $V(0) = V(1)$ for $l < 165$ but not listed all due to the space limit. From Theorem 3, this implies that the number of distinct roots of equation $I(z) = 0$ in $(0, 1]$ is $V(0) - V(1) = 0$. Therefore, $I(z) < 0, z \in (0, 1)$ for $3, \dots, 164$. \square

TABLE I

SIGN CHANGES OF STURM SEQUENCE $I_1(z), \dots, I_m(z)$ OF $I(x)$ IN (5) FOR $l = 3, \dots, 11$. $V(z)$ IS THE NUMBER OF SIGN CHANGES IN THE SEQUENCE.

l	m	$V(z)$	z	$\text{sgn}[I_0(z)], \text{sgn}[I_1(z)], \dots, \text{sgn}[I_m(z)]$
3	13	5	0	---+++-+---++
		5	1	---+++-+---++
4	20	10	0	---+++-+---++
		10	1	---+++-+---++
5	27	12	0	-0+---+++-+---++
		12	1	---+++-+---++
6	33	16	0	-0+---+++-+---++
		16	1	---+++-+---++
7	39	18	0	-0+---+++-+---++
		18	1	---+++-+---++
8	45	22	0	-0+---+++-+---++
		22	1	---+++-+---++
9	51	24	0	-0+---+++-+---++
		24	1	---+++-+---++
10	57	28	0	-0+---+++-+---++
		28	1	---+++-+---++
11	63	30	0	-0+---+++-+---++
		30	1	---+++-+---++

C. Proof of Lemma 3

We first claim that for $z \in (0, 1)$,

If $\frac{al+b}{al+b+1} \in (0, 1)$, then

$$q(z) := z^{al+b}(1-z) \leq \frac{1}{al+b+1}, \quad (8)$$

If $\frac{al+b}{(a+2)l+b-2} \in (0, 1)$, then

$$r(z) := z^{al+b}(1-z^{l-1})^2 \leq \left(\frac{2l-2}{(a+2)l+b-2} \right)^2 \quad (9)$$

Differentiating $q(z)$ gives

$$\frac{dq(z)}{dz} = z^{al+b-1}(al+b-(al+b+1)z).$$

Since $\frac{al+b}{al+b+1} \in (0, 1)$, we see that $z = \frac{al+b}{al+b+1}$ gives the maximum value of $q(z)$.

$$q(z) \leq \left(\frac{al+b}{al+b+1} \right)^{al+b} \frac{1}{al+b+1} < \frac{1}{al+b+1}.$$

Differentiating $r(z)$ gives

$$\frac{dr(z)}{dz} = z^{al+b-1}(1-z^{l-1})((al+b)-((a+2)l+b-2)z^{l-1}).$$

Since $\frac{al+b}{(a+2)l+b-2} \in (0, 1)$, we see that $z = \left(\frac{al+b}{(a+2)l+b-2} \right)^{\frac{1}{l-1}}$ gives the maximum value of $r(z)$. Thus, next inequality holds.

$$r(z) \leq \left(\frac{al+b}{(a+2)l+b-2} \right)^{\frac{al+b}{l-1}} \left(\frac{2l-2}{(a+2)l+b-2} \right)^2 < \left(\frac{2l-2}{(a+2)l+b-2} \right)^2.$$

In (a), we eliminate negative terms except for $-l^3$. Next, in (b), we apply (8) and (9) to each term of (5) by using $l \geq 165$.

We obtain an upper bound of $I(z)$ for $z \in (0, 1)$ as follows.

$$\begin{aligned} I(z) &\stackrel{(a)}{<} -l^3 + 27 \sum_{i=0}^{l-2} z^{3l-2+i}(1-z^{l-1}) \\ &\quad + 108l^2 z^{-3+3l}(1-z^{l-1})^2 \\ &\quad + 9lz^{-4+l}(1-z^{l-1})^2(3z^2 + 16z^{2l} + 8z^{2+l}) \\ &\quad + l^3\{(1-z)z^{-9+l}[56z^{1+5l} + 8z^{3+3l}(13+22z) \\ &\quad + z^{5+l}(41+73z)]\} \\ &\stackrel{(b)}{<} -l^3 + 27 \sum_{i=0}^{l-2} [1] + 108l^2 \left(\frac{2l-2}{5l-5} \right)^2 \\ &\quad + 9l \left(3 \left(\frac{2l-2}{3l-4} \right)^2 + 16 \left(\frac{2l-2}{5l-6} \right)^2 + 8 \left(\frac{2l-2}{4l-4} \right)^2 \right) \\ &\quad + l^3 \left(\frac{56}{6l-7} + \frac{176}{4l-4} + \frac{104}{4l-5} + \frac{41}{2l-3} + \frac{73}{2l-2} \right) \\ &\stackrel{(c)}{<} -l^3 + 27(l-1) + \frac{432l^2}{25} + 9l \left(3 \frac{5}{9} + 16 \frac{1}{5} + 8 \frac{1}{4} \right) \\ &\quad + 5l^3 \left(\frac{59}{29l} + \frac{176}{19l} + \frac{104}{19l} + \frac{73}{9l} + \frac{41}{9l} \right) \\ &< -l^3 + \frac{6775346}{41325}l^2 + \frac{444}{5}l =: \bar{I}(l). \end{aligned}$$

$$\begin{aligned} \left(\frac{2l-2}{3l-4} \right)^2 &\leq \frac{5}{9}, & \left(\frac{2l-2}{5l-6} \right)^2 &\leq \frac{1}{5}, \\ 6l-7 &\geq \frac{29l}{5}, & 4l-4 &\geq \frac{19l}{5}, \\ 4l-5 &\geq \frac{19l}{5}, & 2l-3 &\geq \frac{9l}{5}, \\ 2l-2 &\geq \frac{9l}{5}. \end{aligned}$$

The roots of $\bar{I}(l) = 0$ are 0 and $\frac{3387673 \pm \sqrt{11627977054429}}{41325} \simeq -0.53984, +164.49$. Thus, we conclude that for $l \geq 165$ and $z \in (0, 1)$, $I(z) < \bar{I}(l) < 0$. \square

V. CONCLUSION AND FUTURE WORK

In this paper, we proved that $(l, 3, 3)$ SC-MN codes with $l \geq 3$ achieve capacity on the BEC under BP decoding for sufficiently large L and w . These codes do not have bit nodes of degree two and have low error floors. We proved that the potential threshold and Shannon limit of $(l, r = 3, g = 3)$ MN codes on the BEC are the same.

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APPENDIX

STURM'S THEOREM

Theorem 3 ([8]). *For a polynomial $f(x)$ over \mathbb{R} , we define Sturm sequences $f_i(x)$ ($i = 0, \dots, m$) as $f(x)$, $f'(x)$ and polynomials obtained by applying Euclid's algorithm to $f(x)$ and $f'(x)$.*

$$\begin{aligned} f_0(x) &= f(x), \\ f_1(x) &= f'(x), \\ f_{n-1}(x) &= q_n(x)f_n(x) - f_{n+1}(x) \quad (n = 1, \dots, m-1), \\ f_{m-1}(x) &= q_m(x)f_m(x). \end{aligned}$$

For real number c , let $V(c)$ be the number of sign changes in $f_0(c), f_1(c), \dots, f_m(c)$. If neither $a \in \mathbb{R}$ nor $b \in \mathbb{R}$ is a multiple root of $f(x) = 0$, then the number of distinct roots of $f(x)$ in $(a, b]$ is $V(a) - V(b)$.